The shooting method

1. Suppose that we are solving a boundary-value problem (BVP) that has the boundary conditions $u(2) = 4.2$ and $u(8) = 5.7$. What should the initial-conditions of the first initial-value problem (IVP) be?

Answer: $u(2) = 4.2$ and $u^{(1)}(2) = \frac{5.7 - 4.2}{8 - 2} = 0.25$.

2. Assume that the 2nd-order ordinary differential equation was $u^{(2)}(x) = f(x, u(x), u^{(1)}(x))$. How would you convert this to a system of two first-order IVPs?

Answer:

$w_0^{(1)}(x) = w_1(x)$ $\qquad$ $w_0(2) = 4.2$

$w_1^{(1)}(x) = f(x, w_0(x), w_1(x))$ $\qquad$ $w_1(2) = 0.25$

3. Assume that when find an approximation of the system of two 1st-order IVPs, we find that $w_0(8) \approx 5.34$ and $w_1(8) \approx 6.29$. What slope should we use for our second IVP?

Answer: $w_0(x) = u(x)$, so we should use the first value, so our second initial slope should be

$$u^{(1)}(2) = \frac{2 \cdot 5.7 - 5.34 - 4.2}{8 - 2} = 0.31.$$

4. We change the initial slope for our second system of two 1st-order IVPs. Do we make any change to the ordinary differential equation or the initial value?

Answer: No.

5. Assume that when we find an approximation to this system of two 1st-order IVPs with $w_1(2) = 3.1$, we get that our approximations are $w_0(8) \approx 5.68$ and $w_1(8) \approx 6.42$. What slope should we use for our third iteration?

Answer: We have that (0.25, 5.34) and (0.31, 5.68), and therefore we apply the secant method to the modified problem (0.25, 5.34 – 5.7) and (0.31, 5.6856 – 5.7), which are the points (0.25, –0.36) and (0.31, –0.0144), and thus we apply the secant method, in which case we have that the next slope should be

$$u^{(1)}(2) = \frac{0.25 \cdot (-0.0144) - 0.31(-0.36)}{(-0.0144) - (-0.36)} = 0.3125.$$

6. Does this intuitively make sense?

Answer: Yes. With an initial slope of 0.25, we significantly undershot the value at $x = 8$. Then, with a larger initial slope of 0.31, we still undershot, but only slightly. Thus, we need to find an ever-so-slightly larger initial slope, and the secant method suggests such an initial slope, specifically, 0.3125.

7. Could we use over-relaxation in this case? Recall that with over-relaxation, we chose a value of *w* which was close to 1, but then given a slope s_k and an next slope s_{k+1}, we instead choose an actual next slope equal to $s_{k+1} \leftarrow (1-\omega)s_k + \omega s_{k+1}$.

Answer: Yes, we could use over-relaxation in this case, but as before, we would have to choose an appropriate value of ω. For example, with the first two approximations, we undershot in both cases, and therefore it would make sense to continue using an $\omega \approx 1.05$, whereas if the first overshot and the second undershot, it may be more wise to use an $\omega \approx 0.95$.

8. Suppose you have a 2nd-order differential equation and two boundary conditions $u(1) = 7$ and $u(5) = 4$. You would like to approximate the solution using the shooting method. What should your first slope be?

Answer: –0.75

9. Suppose you used the initial slope you found, but when you approximated the IVP with $u(1) = 7$ and $u^{(1)}(1) = -0.75$, you found that your approximation of the solution to this IVP at $x = 5$ is the value $u(5) = 3$. What should your second slope be?

Answer: –0.5

10. Suppose you used this second slope, but when you approximated the IVP with $u(1) = 7$ and $u^{(1)}(1) = -0.5$, you found that your approximation to the solution to this this IVP at $x = 5$ is the value $u(5) = 4.25$. What should your third slope be?

Answer: –0.55

11. Describe how the secant method is being used to find the value in Question 10.

Answer: You should answer this yourself.

12. In Question 10, it happened that we were finding the root of the linear polynomial interpolating the two points (–0.75, –1) and (–0.5, 0.25). This, in a sense, the bracketed secant method, but is guaranteed that we are using the bracketed method?

Answer: No, for if in Question 10, we found that using the initial slope –0.5, the solution is $u(5) = 3.8$, then we are not using the bracketed secant method.

13. Suppose we have the same result in Question 9, but then instead of Question 10, instead, we found that at $x = 5$, the value $u(5) = 3.8$. What should your third slope be?

Answer: –0.4375

14. Is there any reason, given three initial slopes and three values at $u(5)$ that we could not find a quadratic interpolating polynomial and find a root of that quadratic?

Answer: No, only now we must make sure that 1) we are choosing the correct root, and 2) we must deal with the potential issue that the interpolating polynomial has no root, in which case, we should fall back on the interpolating linear polynomial on the last two points.

15. Suppose you had the following two boundary conditions: $u(1) = 7$ and $u^{(1)}(5) = 1$. How would you modify the shooting method to account for one Dirichlet (fixed) boundary condition $u(1) = 7$ and one Neumann (fixed slope) boundary condition $u^{(1)}(5) = 1$?

Answer: If we have no more information, we might as well start with the IVP with $u^{(1)}(1) = 1$. We would then approximate the solution and see what the derivative is at $u(5)$.